Modeling Fuzzy Topological Predicates for Fuzzy Regions

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ABSTRACT

Spatial database systems and Geographical Information Systems (GIS) are currently only able to handle *crisp spatial objects*, i.e., objects whose extent, shape, and boundary are precisely determined. However, GIS applications are also interested in managing *vague* or *fuzzy spatial objects*. *Spatial fuzziness* captures the inherent property of many spatial objects in reality that do not have sharp boundaries and interiors or whose boundaries and interiors cannot be precisely determined. While topological relationships have been broadly explored for crisp spatial objects, this is not the case for fuzzy spatial objects. In this paper, we propose a novel model to formally define fuzzy topological predicates for simple and complex fuzzy regions. The model encompasses six fuzzy predicates (overlap, disjoint, inside, contains, equal and meet), wherein here we focus on the *fuzzy overlap* and the *fuzzy disjoint* predicates only. For their computation we consider two low-level measures, the degree of membership and the degree of coverage, and map them to high-level fuzzy modifiers and linguistic values respectively that are deployed in spatial queries by end-users.

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1. INTRODUCTION

Increasingly, geoscientists are interested in modeling spatial phenomena that are characterized by the feature of *spatial fuzziness*. Spatial fuzziness captures the inherent property of many spatial objects in reality that do not have exact locations, strict boundaries, and sharp interiors. However, spatial database systems and Geographical Information Systems (GIS) are currently only able to handle crisp spatial objects, such as points, lines and regions. These spatial objects are characterized by an exact location and a precisely defined extent, shape, and boundary in space and hence cannot adequately represent spatial fuzziness. In the geoscience and GIS domains, fuzzy set theory has become a popular tool for modeling such *vague* or *fuzzy spatial objects*. Spatial data types for fuzzy points, fuzzy lines, and fuzzy regions have been provided for representing them. The central idea is to relax the strict decision of belonging (value 1) or non-belonging (value 0) of an element to a set. Instead, partial membership is allowed and expressed by a value in the interval $[0, 1]$.

As for the crisp spatial objects that have spatial operations (e.g., topological predicates, and geometric set operations), a number of fuzzy spatial operations has been defined to handle the fuzzy spatial objects, like fuzzy geometric set operations (e.g., fuzzy geometric union) and fuzzy numerical operations (e.g., fuzzy area). While topological relationships have been largely explored on crisp spatial objects, this is not the case for fuzzy spatial objects.

In this paper, we focus on this gap in the literature. We propose a novel model to formally define fuzzy topological predicates for simple and complex fuzzy regions. The whole model encompasses six predicates named *fuzzy overlap*, *fuzzy disjoint*, *fuzzy inside*, *fuzzy contains*, *fuzzy equal*, and *fuzzy meet*. The advantages of the model can be described as follows. It handles simple and complex fuzzy spatial regions. It also provides fuzzy modifiers (e.g. *slightly*) and linguistic values (e.g. *small*) that can be deployed in spatial queries by end-users. Furthermore, it avoids the drawback of the use of the supremum membership degree concept (see Section 2). Due to space limitations, we here focus only on the description and use of the *fuzzy overlap* and the *fuzzy disjoint* predicates.

This paper is organized as follows. Section 2 surveys re-

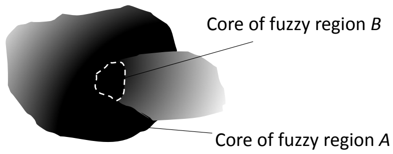

Figure 1: The problem of the supremum membership degree for the fuzzy overlap between two fuzzy regions. The core of the fuzzy region B is highlighted by a dashed white line. Darker areas indicate higher membership degrees than lighter areas.

lated work. Section 3 presents the *fuzzy overlap* and the *fuzzy disjoint* predicates. Section 4 shows how to transform these fuzzy predicates into Boolean predicates. Section 5 concludes the paper and presents future work.

2. RELATED WORK

In general, the available approaches that model *fuzzy topological predicates for fuzzy regions* [1, 2, 6] have several problems in the practicable applicability and in the support provided for the end-user in spatial queries. Firstly, the majority of the approaches only deals with simple fuzzy regions [1, 2], while our proposed model, in the same way as [6], deals with both simple and complex fuzzy regions. Using only simple fuzzy regions restricts the expressiveness, since real-world phenomena commonly have a complex structure. Secondly, only few approaches [1] provide fuzzy modifiers that can be deployed in spatial queries by end-users. Our proposed model also use this concept, which transforms the non-intuitive and quantitative fuzzy values in the interval $[0, 1]$ provided by the two low-level measures into intuitive, fuzzy, qualitative, and equivalent terms that have significance for the end-user. The two low-level measure considered in our model are the degree of membership and the degree of coverage (e.g. the coverage of overlapping area or distance). The degree of coverage is not considered in the available approaches.

The most important drawback of some available approaches [2, 6] refers to the computation of the supremum membership degree of the fuzzy geometric intersection between two fuzzy regions. It does not consider *all* points of the overlapping area and does hence not produce realistic results. Figure 1 illustrates the problem. In this figure, the supremum membership degree is 1 since there are points in the overlapping area that have the membership value 1 in both fuzzy regions. However, it is not sure that the overall degree of overlapping is 1, since the fuzzy part (i.e., all points with membership degree less than 1) of the fuzzy region B overlaps the core (i.e., all points with membership degree equal to 1) of the fuzzy region A. Therefore, we cannot state that there is a fuzzy overlap with a membership degree equal to 1. Note that this is different from the crisp case, where the overlapped area of the crisp overlap is composed only by crisp points with membership values equal to 1. To avoid this drawback, our proposed model is *not* based on the supremum membership degree.

3. FUZZY TOPOLOGICAL PREDICATES

We first define the concept of fuzzy region object (*fregion*), since it is the basis for fuzzy topological predicates. Intuitively, it has a similar areal geometry as a crisp region object [5] but it may have a vague boundary and/or a vague interior. A *fregion* object is represented by a fuzzy point set $\tilde{A}$ in the plane with a membership function $\mu_{\tilde{A}} : \mathbb{R}^2 \to [0, 1]$ (with particular properties discussed in [3]).

In this paper, we propose a new ad hoc approach that takes a holistic view, that is, it does not take in consideration the distinction of components of fuzzy region objects. It is based on the two low-level quantitative measures of *degree of membership* and *degree of area or distance coverage.*

With respect to the degree of membership, our proposed model considers the weighted membership degrees of *all* intersection points of two fuzzy region objects. This leads to a rather different design of fuzzy topological predicates and has a positive impact on their correctness and suitability in practice. For a predicate P, we use a fuzzy predicate P_1 : *fregion* $\times$ *fregion* $\to [0, 1]$ to determine the degree of membership of P.

With respect to the degree of area or distance coverage, our model incorporates the ratio of the intersecting area of two fuzzy region objects to their total area as well as the ratio of their minimum distance to the maximum distance in their finite universe of discourse. For P, we use a fuzzy predicate P_2 : *fregion* $\times$ *fregion* $\to [0, 1]$ to determine the degree of area or distance coverage. Let $\tilde{A}, \tilde{B} \in$ *fregion*. We define a fuzzy topological predicate P as

$$P(\tilde{A}, \tilde{B}) = (P_1(\tilde{A}, \tilde{B}), P_2(\tilde{A}, \tilde{B}))$$

This means that $P(\tilde{A}, \tilde{B}) \in [0, 1] \times [0, 1]$ since we consider two different degree criteria and thus combine two fuzzy predicates into a single predicate. In case that a fuzzy topological predicate is applied to two crisp region objects, we ensure that the same value (out of $\{0, 1\}$) is returned as the corresponding crisp topological predicate would return.

In the following subsections, we formally define the predicates P_1 and P_2 for each fuzzy topological predicate $P \in \{foverlap, fdisjoint\}$. The remaining of fuzzy topological predicates can be derived from these design considerations. For each predicate we specify when it yields 0, 1, or a value of the interval $]0, 1[$. For this purpose, we will make use of some further crisp and fuzzy spatial concepts. The overloaded operations $\otimes$ (geometric intersection) and $\oplus$ (geometric union) of two crisp spatial objects [5] and two fuzzy spatial objects [3] respectively. We also use cluster topological predicates on crisp spatial data types, $overlap_c$, $equal_c$, $disjoint_c$ and $meet_c$ [5]. We also employ the metric operations *area* and *dist* to compute the area of a crisp region object and the minimum distance between two crisp objects. Similarly, we use the *farea* and *fdist* [2] to determine the area of a fuzzy region object and the minimum distance between two fuzzy regions. Finally, we also use fuzzy set operations (e.g. fuzzy set containment) and operations that maps fuzzy sets to crisp sets, such as *core* and *supp* for the support [7].

3.1 Fuzzy Overlap

The fuzzy topological predicate *foverlap* evaluates the two fuzzy predicates $foverlap_1$ and $foverlap_2$. The fuzzy predicate $foverlap_1$ determines the degree of overlapping of two fuzzy region objects $\tilde{A}$ and $\tilde{B}$. Two conditions must hold in order to be definitely sure that $\tilde{A}$ and $\tilde{B}$ overlap and thus $foverlap_1(\tilde{A}, \tilde{B}) = 1$ holds. The first condition is that their crisp supports overlap. The second condition is that all points of the overlapping area have the membership value 1, that is, they do all definitely belong to the intersection and form a crisp area (i.e. its support is equal to its core).

Two conditions must alternatively hold so that $\tilde{A}$ and $\tilde{B}$ do definitely not overlap, that is, $foverlap_1(\tilde{A}, \tilde{B}) = 0$ holds. The first condition is that there is no geometric intersection of the supports of $\tilde{A}$ and $\tilde{B}$. This means that the two supports are disjoint or adjacent (i.e meet). The second alternative condition relates to a fuzzy set containment situation between $\tilde{A}$ and $\tilde{B}$ where $\tilde{A}$ is contained in $\tilde{B}$, or $\tilde{B}$ is contained in $\tilde{A}$, or $\tilde{A}$ and $\tilde{B}$ are equal. Note that a set containment requirement between the supports of $\tilde{A}$ and $\tilde{B}$ is a too weak condition, since if $supp(\tilde{A}) \subseteq supp(\tilde{B})$ and $\tilde{A} \not\subseteq \tilde{B}$ holds, then there are points that have a higher membership value in $\tilde{A}$ than in $\tilde{B}$, and thus a fuzzy overlap situation.

The remaining case is that $foverlap_1(\tilde{A}, \tilde{B}) \in\ ]0, 1[$ holds. This situation can be visualized as follows. Each graph of a membership function of a fuzzy region object determines a (complex) volume below this graph. We have a fuzzy overlap situation if the corresponding membership function volumes of the two fuzzy region objects intersect in a (complex) volume. For the computation of the degree of overlapping of $\tilde{A}$ and $\tilde{B}$, we first have to compute their fuzzy intersection, which leads to a fuzzy region object, and then form the ratio of its fuzzy area with the crisp area of its support. This ratio is a value in the interval $]0, 1[$, and as discussed previously, Figure 1 shows a fuzzy overlap situation that returns such a value for $foverlap_1$.

The fuzzy predicate $foverlap_2$ determines the degree of area coverage of two fuzzy region objects $\tilde{A}$ and $\tilde{B}$. If we have a fuzzy overlap situation from a membership degree perspective ($foverlap_1(\tilde{A}, \tilde{B}) > 0$), we compute the ratio of the fuzzy area of their intersection with the fuzzy area of their union. Otherwise, we return the value 0.

We are now able to provide a formal definition of the fuzzy topological predicate *foverlap*. Let $\tilde{A}, \tilde{B} \in fregion$. Then

$$foverlap(\tilde{A}, \tilde{B}) = (foverlap_1(\tilde{A}, \tilde{B}), foverlap_2(\tilde{A}, \tilde{B}))$$

where

$$foverlap_1(\tilde{A}, \tilde{B}) = \begin{cases} 1 & \text{if } overlap_c(supp(\tilde{A}), supp(\tilde{B})) \wedge \\ & \quad equal_c(core(\tilde{A} \otimes \tilde{B}), supp(\tilde{A} \otimes \tilde{B})) \\ 0 & \text{if } disjoint_c(supp(\tilde{A}), supp(\tilde{B})) \vee \\ & \quad meet_c(supp(\tilde{A}), supp(\tilde{B})) \vee \\ & \quad \tilde{B} \subseteq \tilde{A} \vee \tilde{A} \subseteq \tilde{B} \\ \frac{farea(\tilde{A} \otimes \tilde{B})}{area(supp(\tilde{A} \otimes \tilde{B}))} & \text{otherwise} \end{cases}$$

and

$$foverlap_2(\tilde{A}, \tilde{B}) = \begin{cases} 0 & \text{if } foverlap_1(\tilde{A}, \tilde{B}) = 0 \\ \frac{farea(\tilde{A} \otimes \tilde{B})}{farea(\tilde{A} \oplus \tilde{B})} & \text{otherwise} \end{cases}$$

This definition reveals some interesting aspects and advantages. First, the property holds that $foverlap_1(\tilde{A}, \tilde{B}) = 0 \Leftrightarrow foverlap_2(\tilde{A}, \tilde{B}) = 0$. The "$\Rightarrow$" direction is given by the first case of $foverlap_2$. The "$\Leftarrow$" direction can be derived from the observation that if there is no areal intersection between $\tilde{A}$ and $\tilde{B}$, the geometric intersection of their supports must be empty, which is covered by the second case of $foverlap_1$. Second, always $foverlap_2(\tilde{A}, \tilde{B}) \in [0, 1[$ holds. If $foverlap_2(\tilde{A}, \tilde{B})$ could return the value 1, this would mean that $\tilde{A}$ and $\tilde{B}$ are equal. However, then $foverlap_1(\tilde{A}, \tilde{B})$ would return 0 and consequently $foverlap_2(\tilde{A}, \tilde{B})$ would return 0 as well, which leads to a contradiction. Third, our approach takes all intersected points into account. This is superior to the "supremum" approach, which prioritizes those points of the overlapping area of $\tilde{A}$ and $\tilde{B}$ with the maximum membership value and ignores all other points. Therefore, many overlap situations with the same supremum value cannot be differentiated and lead to the same degree of overlapping that is often counterintuitive and distorting the reality.

3.2 Fuzzy Disjoint

The fuzzy topological predicate *fdisjoint* evaluates the two fuzzy predicates $fdisjoint_1$ and $fdisjoint_2$. The fuzzy predicate $fdisjoint_1$ determines the degree of disjointedness of two fuzzy region objects $\tilde{A}$ and $\tilde{B}$. The only condition to be definitely sure that $\tilde{A}$ and $\tilde{B}$ are disjoint and thus $fdisjoint_1(\tilde{A}, \tilde{B}) = 1$ holds is that their supports are disjoint. Therefore, the fuzzy predicate $fdisjoint_1$ excludes the fuzzy predicate $foverlap_1$ since the first condition of this predicate to return 0 is the disjointedness of their supports.

Two conditions must alternatively hold so that $\tilde{A}$ and $\tilde{B}$ are definitely not disjoint, that is, $fdisjoint_1(\tilde{A}, \tilde{B}) = 0$ holds. The first condition is that $\tilde{A}$ and $\tilde{B}$ definitely overlap. The second alternative condition relates to a fuzzy set containment situation between $\tilde{A}$ and $\tilde{B}$ (i.e. $\tilde{A}$ is contained in $\tilde{B}$, or $\tilde{B}$ is contained in $\tilde{A}$, or $\tilde{A}$ and $\tilde{B}$ are equal).

The remaining case is that $fdisjoint_1(\tilde{A}, \tilde{B}) \in\ ]0, 1[$ holds. As in the cases where $fdisjoint_1(\tilde{A}, \tilde{B}) \in \{0, 1\}$, it is complementary to the $foverlap_1$ situation. In this case, we take the degree of $foverlap_1$ as a basis to compute its complement.

The fuzzy predicate $fdisjoint_2$ determines the degree of distance coverage of two fuzzy region objects $\tilde{A}$ and $\tilde{B}$. If we have a fuzzy disjoint situation from a membership degree perspective ($fdisjoint_1(\tilde{A}, \tilde{B}) > 0$), we compute the proportion of the fuzzy distance and the maximum distance of the selected universe of discourse. The maximum distance is the Euclidean distance of a predefined rectangle with p and q as its lower left and upper right vertices respectively. If there is no fuzzy disjoint situation, we return the value 0.

We are now able to provide a formal definition of the fuzzy topological predicate *fdisjoint*. Let $\tilde{A}, \tilde{B} \in fregion$. Then

$$fdisjoint(\tilde{A}, \tilde{B}) = (fdisjoint_1(\tilde{A}, \tilde{B}), fdisjoint_2(\tilde{A}, \tilde{B}))$$

where

$$fdisjoint_1(\tilde{A}, \tilde{B}) = \begin{cases} 1 & \text{if } disjoint_c(supp(\tilde{A}), supp(\tilde{B})) \\ 0 & \text{if } foverlap_1(\tilde{A}, \tilde{B}) = 1 \vee \\ & \quad \tilde{B} \subseteq \tilde{A} \vee \tilde{A} \subseteq \tilde{B} \\ 1 - foverlap_1(\tilde{A}, \tilde{B}) & \text{otherwise} \end{cases}$$

and

$$fdisjoint_2(\tilde{A}, \tilde{B}) = \begin{cases} 0 & \text{if } fdisjoint_1(\tilde{A}, \tilde{B}) = 0 \\ \frac{fdist(\tilde{A}, \tilde{B})}{dist(p, q)} & \text{otherwise} \end{cases}$$

Similarly as for the predicate *foverlap*, we can show that $fdisjoint_1(\tilde{A}, \tilde{B}) = 0 \Leftrightarrow fdisjoint_2(\tilde{A}, \tilde{B}) = 0$ holds. Additionally, $disjoint_2(\tilde{A}, \tilde{B}) \in [0, 1[$ holds. This predicate cannot return the value 1 since $\tilde{A}$ and $\tilde{B}$ would degenerate into fuzzy points with the supports p and q.

4. DEPLOYING FUZZY TOPOLOGICAL PREDICATES IN SPATIAL QUERIES

Aiming to transform the non-intuitive and quantitative fuzzy predicate values in the interval [0, 1] into intuitive, fuzzy, and qualitatively equivalent terms that have significance for the end-user, the proposed model: (i) maps the degree of membership to *fuzzy modifiers*; and (ii) maps the degree of coverage to *linguistic values*. In this section, we show how to perform these mappings.

We use the notion of the *trapezoidal fuzzy set* to map low-level measures to high-level concepts. Let a, b, c, d $\in \mathbb{R}$. Then a trapezoidal fuzzy set $\tilde{T}$ is defined as a tuple (a, b, c, d), denoting that a low-level measure increases linearly from a to b, is constant and equal to 1 from b to c, and decreases linearly from c to d. For each low-level measure, a set of trapezoidal fuzzy sets is created to classify a value in $[0, 1]$. Each classification, which is represented by a qualitative linguistic description, is in the form (a, b, c, d). We assume that the qualitative linguistic descriptions are either predefined in the spatial query language or user-defined.

For the degree of membership, the qualitative linguistic descriptions are represented as *fuzzy modifiers*. For example, for a given application domain, an end-user could specify a high-level classification for the degree of membership that has seven fuzzy modifiers, *a little bit* = (0, 0, 0.03, 0.08), *somewhat* = (0.03, 0.08, 0.2, 0.26), *slightly* = (0.2, 0.26, 0.39, 0.45), *averagely* = (0.39, 0.45, 0.62, 0.69), *largely* = (0.62, 0.69, 0.82, 0.89), *mostly* = (0.82, 0.89, 0.93, 0.95), and *quite* = (0.93, 0.95, 1, 1).

For the degree of coverage, the qualitative linguistic descriptions are represented as *linguistic values*. For instance, a spatial query language can predefine a high-level classification for the degree of coverage that has five linguistic values, *tiny* = (0, 0, 0.07, 0.09), *small* = (0.07, 0.09, 0.35, 0.45), *medium* = (0.35, 0.45, 0.65, 0.75), *large* = (0.65, 0.75, 0.87, 0.92), and *huge* = (0.87, 0.92, 1, 1).

Aiming to allow that the high-level concepts can be used in a spatial query language as a crisp Boolean predicate, we propose to define a function for each fuzzy topological predicate, which returns a crisp Boolean value (true or false). Let $\tilde{A}, \tilde{B} \in \mathit{fregion}$. We specify a fuzzy topological predicate FP in a spatial query language as

$$FP(\tilde{A}, \tilde{B}, C_1(P_1(\tilde{A}, \tilde{B})), C_2(P_2(\tilde{A}, \tilde{B}))) \rightarrow \{\mathit{false}, \mathit{true}\}$$

C_1 is a function to perform a classification for the degree of membership, and C_2 is a function to perform a classification for the degree of coverage. These functions take as parameters the degree of membership and the degree of coverage and then return its respective classification according the trapezoidal fuzzy sets (e.g. the value 0.8 for the degree of coverage is classified as *large*). For example, $\mathit{foverlap}(\tilde{A}, \tilde{B}, \mathit{slightly}, \mathit{large})$ returns *true* when "$\tilde{A}$ *slightly* overlaps $\tilde{B}$ and the overlapping area is *large*".

The definitions presented in this section allow the definition of SQL statements with fuzzy topological predicates. For instance, considering the attribute *geo* of type *fregion* in two table schemas *fish* and *lake*, a query to return only the species name whose its habitat area *averagely* overlap with a hunting area and its overlapping area is *huge* can be written as

```
SELECT S.name
FROM   species S, hunting H
WHERE  foverlap(S.geo, H.geo, averagely, huge)
```

5. CONCLUSIONS AND FUTURE WORK

This paper provides a new approach to modeling the fuzzy topological predicates *fuzzy overlap* and *fuzzy disjoint* for fuzzy regions from low-level quantitative measures to high-level fuzzy modifiers and linguistic values for end-users. The main advantages of our approach are that (i) it can handle complex fuzzy spatial objects, (ii) all intersection points of two fuzzy regions are considered for evaluation so that realistic fuzzy topological predicate results can be obtained, (iii) fuzzy topological predicates yielding values in the interval $[0, 1]$ can be mapped to suitable Boolean topological predicates by deploying the concepts of fuzzy modifiers (e.g. *slightly*) and linguistic values (e.g. *small*), and (iv) these Boolean predicates can be embedded in spatial queries.

Future work will deal with the definition for the remaining fuzzy topological predicates, such as *fuzzy inside*, *fuzzy contains*, *fuzzy equal* and *fuzzy meet*. Additionally, their implementation and their mappings to Boolean topological predicates will be studied as well. Our goal is to perform the implementation in the context of the *Spatial Plateau Algebra* [4], which provides a general concept for the implementation of fuzzy spatial data types.

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